§ Review of Linear algebra :

Let $A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \in Mat_{2\times 2}$ (2×2 matrices).

$A : \mathbb{R}^2 \to \mathbb{R}^2$ by $\vec{v} \to A\vec{v}$ $\quad A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{21}v_2 \\ a_{12}v_1 + a_{22}v_2 \end{pmatrix}$

This is a linear transformation

$$A(\vec{v} + c\vec{w}) = A\vec{v} + cA\vec{w}$$

Basis of a vector space :

If we consider V : a vector space , a basis of V is a set $\{v_1, \dots, v_n\}$ satisfying

1) for any $x \in V$, $x = \sum_{i=1}^{n} \alpha_i v_i$ for some $\alpha_1 \dots \alpha_n$.

2) $\sum_{i=1}^{n} \alpha_i v_i = 0$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

For example, if we have $V = \mathbb{R}^2$

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis of V.

$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is also a basis

$n = \dim(V)$

Now, if $\{v_1\ v_2\}$ is a basis of $\mathbb{R}^2$, then

$$\det\begin{pmatrix} v_1 & v_2 \end{pmatrix} \neq 0.$$

$$\left(\text{or } \begin{pmatrix} v_1 v_2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0 \right)$$

[If $\det(v_1\ v_2) = 0 \Rightarrow \exists\ \alpha_1\ \alpha_2$ s.t $\alpha_1 v_1 + \alpha_2 v_2 = 0$]

Conversely, if $\det(v_1\ v_2) \neq 0 \Rightarrow \{v_1\ v_2\}$ is a basis.

Eigenvalues and Eigenvectors:

Let $A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$, $\det(A - xI) = (a_{11} - x)(a_{22} - x) - a_{12}a_{21}$

$$= x^2 - (a_{11} + a_{22})x + \begin{pmatrix} a_{11}a_{22} \\ - a_{12}a_{21} \end{pmatrix}$$

$$= x^2 - tr(A)x + \det(A) \quad \text{---} (*)$$

We call the solutions of $(*)$ the eigenvalues of A.

Example: if $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \Rightarrow x^2 - 6x + 5 = 0 \quad x = 5 \text{ or } 1$
eigenvalues.

Thm1: If λ is an eigenvector of A, then there exists $\vec{v}$ s.t

$$Av = \lambda v$$

Thm2: If A is symmetric, then $\exists\ M$ s.t. $M^T A M = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
for $\lambda_1\ \lambda_2$ are eigenvalues of A.

Proof of Thm 1:

$$\det(A-\lambda I)=0 \Leftrightarrow \exists(\alpha_1,\alpha_2):=\vec{v} \ \text{s.t} \ (A-\lambda I)\vec{v}=0$$
$$\Leftrightarrow A\vec{v}=\lambda\vec{v}$$

Proof of Thm 2: Let $\lambda_1 \lambda_2$ are the eigenvalues of A.

If $\lambda_1 \neq \lambda_2$, $\exists \ \vec{v}_1, \vec{v}_2$ s.t $\begin{matrix} A\vec{v}_1 = \lambda_1 \vec{v}_1 \\ A\vec{v}_2 = \lambda_2 \vec{v}_2 \end{matrix}$ (we can take $|\vec{v}_1|=|\vec{v}_2|=1$)

taking $M=\left(\vec{v}_1, \vec{v}_2\right)$, we have

$$AM = M\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Meanwhile, since $\lambda_1\vec{v}_1\cdot\vec{v}_2 = A\vec{v}_1\cdot\vec{v}_2 = \vec{v}_1\cdot A\vec{v}_2$
$= \lambda_2 \vec{v}_1\cdot\vec{v}_2$

$$\Rightarrow (\lambda_1-\lambda_2)\vec{v}_1\cdot\vec{v}_2=0. \text{ So } M^T=M^{-1}$$

$$\Rightarrow M^TAM=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

If $\lambda_1=\lambda_2$, then $A=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$.

This is because $\begin{cases} \lambda_1^2 = a_{11}a_{22}-a_{12}^2 \\ 2\lambda_1 = a_{11}+a_{22} \end{cases} \Rightarrow \begin{cases} 4\lambda_1^2 = 4a_{11}a_{22}-4a_{12}^2 \\ 4\lambda_1^2 = a_{11}^2+a_{22}^2+2a_{11}a_{22} \end{cases}$

$$\Rightarrow (a_{11}-a_{22})^2+4a_{12}^2=0$$

$$\therefore a_{11}=a_{22}, \quad a_{12}=0.$$

In general, any matrix A with distinct eigenvalues can be diagonalized.

$$M^{-1}AM = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with $M = \left(\vec{v}_1 , \vec{v}_2 \right)$, $A\vec{v}_1 = \lambda_1 \vec{v}_1$. $A\vec{v}_2 = \lambda_2 \vec{v}_2$.

If $\lambda_1 = \lambda_2$, $\exists M$ s.t

$$M^{-1}AM = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

<u>Jordan form</u>.

§ First order ODE System:

Let $r(t) = (x(t), y(t))$ be a plane curve

Suppose that this curve satisfies the following rule:

$$\begin{cases} r'(t) = (f(t), g(t)) & \text{for some } f(t), g(t). \\ r(0) = (x_0, y_0). \longrightarrow \text{initial condition} \end{cases}$$

Then we can solve $r(t)$ by FTC:

$$\begin{cases} x(t) = \int_0^t f(s)\,ds + x_0 \\ y(t) = \int_0^t g(s)\,ds + y_0 \end{cases}$$

In general, we can consider a equation

$$\begin{cases} x'(t) = f(x, y, t) \\ y'(t) = g(x, y, t) \end{cases}$$

This is called a planar system of ordinary differential equation.

<u>Linear equation:</u>

Let A be a matrix. $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$(*) \quad \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$ is a linear system of ODE.

If $r_1(t)$ $r_2(t)$ are solution of $(*)$, then

$\alpha r_1 + \beta r_2$ is also a solution of $(*)$.

$$\begin{cases} r_1'(t) = A r_1(t) \\ r_2'(t) = A r_2(t) \end{cases} \Rightarrow \alpha r_1'(t) + \beta r_2'(t) = \alpha A r_1(t) + \beta A r_2(t)$$
$$= A(\alpha r_1 + \beta r_2).$$

Now, we consider a linear system with an initial condition.

$$\begin{cases} r'(t) = A r(t) \\ r(0) = (x_0\ y_0). \end{cases}$$

Suppose that A is a diagonal matrix. $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Then we have

$$\begin{cases} x'(t) = \lambda_1 x(t) \\ y'(t) = \lambda_2 y(t) \\ (x(0), y(0)) = (x_0, y_0) \end{cases}$$

Now, notice that $e^{-\lambda_1 t} \cdot x'(t) - e^{-\lambda_1 t} \lambda_1 x(t)$

$$= \frac{d}{dt}\left(e^{-\lambda_1(t)} x(t)\right) = 0.$$

So $e^{-\lambda_1(t)} X(t) = \text{const.}$

Take $t=0$, we have $e^0 X(0) = X(0) = x_0$,

$$\text{so } e^{-\lambda_1 t} X(t) = x_0 \Rightarrow X(t) = x_0 e^{\lambda_1 t}$$

Similarly, $y(t) = y_0 e^{\lambda_2 t}$

So we have the solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{\lambda_1 t} \\ y_0 e^{\lambda_2 t} \end{pmatrix}$

Now, suppose A is <u>diagonalizable</u>. Then $\exists M$ s.t

$$M^{-1} A M = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Taking $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = M^{-1} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = M^{-1} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = M^{-1} A . \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = M^{-1} A M \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 u(t) \\ \lambda_2 v(t) \end{pmatrix}$$

Meanwhile, $\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = M^{-1} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = M^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

So we can solve $(x(t), y(t))$.

Example: Let $A = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$. Solve

$$\begin{cases} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{cases}$$

Sol: $\det(A - xI) = (2-x)(1-x) - 2$

$$= x^2 - 3x = 0$$

$\therefore \lambda_1 = 3, \ \lambda_2 = 0$

$$A - \lambda_1 I = \begin{pmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}, \quad \vec{V_1} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

$$A - \lambda_2 I = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}, \quad \vec{V_2} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}$$

$$M = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} \quad M^T = M^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

So $M^{-1} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}$ has solution:

$$\begin{cases} u(t) = u_0 e^{3t} \\ v(t) = v_0 \cdot e^{0t} = v_0 \end{cases} \quad \text{where } \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = M^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}+1}{\sqrt{3}} \\ \frac{\sqrt{2}-1}{\sqrt{3}} \end{pmatrix}$$

So $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}+1}{\sqrt{3}} \cdot e^{3t} \\ \frac{\sqrt{2}-1}{\sqrt{3}} \end{pmatrix}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = M \cdot \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{2}+1}{\sqrt{3}} e^{3t} \\ \frac{\sqrt{2}-1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2+\sqrt{2}}{3} e^{3t} - \frac{\sqrt{2}-1}{3} \\ \frac{\sqrt{2}+1}{3} e^{3t} + \frac{2-\sqrt{2}}{3} \end{pmatrix}$$

<u>Generalization</u>: Consider

$$\begin{cases} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \\ \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \end{cases}$$

where A, $f(t)$, $g(t)$, x_0 y_0 are given. A diagonalizable.

To solve this eq. let $M^{-1} A M = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

$$M^{-1} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} := \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad M^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} := \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

$$M^{-1} \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} := \begin{pmatrix} h(t) \\ k(t) \end{pmatrix}$$

So

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 u(t) \\ \lambda_2 v(t) \end{pmatrix} + \begin{pmatrix} h(t) \\ k(t) \end{pmatrix}$$

$$\Rightarrow \quad e^{-\lambda_1 t} u'(t) - \lambda_1 e^{-\lambda_1 t} u(t) = e^{-\lambda_1 t} h(t)$$

$$= \frac{d}{dt}\left(e^{-\lambda_1 t} u(t)\right)$$

$$e^{-\lambda_1 t} u(t) = \int_0^t e^{-\lambda_1 s} h(s)\, ds + u_0$$

$$\Rightarrow \quad u(t) = e^{\lambda_1 t} \int_0^t e^{-\lambda_1 s} h(s)\, ds + u_0 e^{\lambda_1 t}.$$

Similarly $\quad v(t) = e^{\lambda_2 t} \int_0^t e^{-\lambda_2 s} k(s)\, ds + v_0 e^{\lambda_2 t}.$

and $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = M \cdot \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}.$